

Rigid particle revisited: extrinsic curvature yields the Dirac equation

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We analyze in details the quantization procedure for relativistic particle with higher-derivative term linear on the first extrinsic curvature (rigidity). We find that, contrary to common opinion, its quantization in terms of $so(3,2)$ -algebra yields massive Dirac equation.

I. INTRODUCTION

At the end of eighties much attention has been paid to the study of relativistic particle models with Lagrangians depending on extrinsic curvatures of the world line. These studies were mostly inspired by the Polyakov's papers on rigid strings [1] and Chern-Simons theories [2]. Initially these models were considered as toy models for the above mentioned field-theoretical ones, but very soon it was realized that they are of their own interest, and probably could be considered as mechanical models of relativistic spin (see, e.g. [3]-[10]). The first system of this sort has been suggested by Pisarski [3] as a toy model of rigid string. It was given by the action

$$S = \int d\tau \sqrt{-\dot{x}^2} [-mc + c_1 k_1(\dot{x}, \ddot{x})], \quad (1)$$

where k_1 is the first extrinsic curvature of world line

$$k_1 = \frac{\sqrt{-(\dot{x}^2 \ddot{x}^2 - (\dot{x} \ddot{x})^2)}}{|\dot{x}|^3}. \quad (2)$$

This system, as well as other three- and four-dimensional systems with the Lagrangians depending on extrinsic curvatures were investigated by many authors. Key observations there were done by M. S. Plyushchay. In particular, he found that when $m = 0$, it describes massless particle with the helicity c_1 [5], while the case $m \neq 0$, $c_1 \neq 0$ implies a model with eight-dimensional phase space [4], which on the quantum level does not describe an elementary system (described by irreducible representation of Poincaré group). Further studies of similar systems in arbitrary space-time [8], as well as on null-curves [10] result in the same conclusion: only the actions proportional to single extrinsic curvature yield, upon quantization, the massless irreducible representations of Poincaré group. Besides it was established that the mass term in (1) prohibits the constraint which could be classical analog of Klein-Gordon equation.

However, Klein-Gordon equation could follow from the Dirac one at the *quantum* level, so that it is not

necessary to have an analog of the later equation at the classical level. For the discussion of these issues we refer to [11, 12], where the model of Dirac equation has been suggested (that is mechanics which has classical analog of the Dirac equation, but does not imply analog of Klein-Gordon equation), and to [13] for the model of Dirac electron (that is mechanics which implies classical analogs of both Dirac and Klein-Gordon equations).

In the present note, following this ideology, we analyze the action (1) with $m \neq 0$ and $c_1 = \frac{\sqrt{3}\hbar}{2}$, and show that its canonical quantization leads to the Dirac equation with the mass

$$M = \frac{\sqrt{3}}{2}m. \quad (3)$$

Any other choice of c_1 turns out to be inconsistent with our quantization procedure, see below. So, in contrast with common opinion, quantization of the model of "rigid particle" given by the action (1) results in the elementary system of spin one-half.

II. HAMILTONIAN FORMULATION

Consider the time-like curve $x^\mu(\tau)$ (parameterized by arbitrary τ) in four-dimensional Minkowski space $\eta^{\mu\nu} = (-, +, +, +)$: $\dot{x}^2 < 0$.

Let us consider the action (1), denoting, for convenience, $-mc = c_0$. To be able to construct Hamiltonian formulation of the theory, we first use the Ostrogradsky method and represent our higher-derivative Lagrangian as a Lagrangian with first-order derivatives. That is we represent the action (1) in the form

$$S = \int d\tau \left[c_0 |\omega| + c_1 \frac{\sqrt{\dot{\omega} N(\omega) \dot{\omega}}}{|\omega|} + p(\dot{x} - \omega) \right], \quad (4)$$

where we have introduced the projector

$$N^{\mu\nu} = \eta^{\mu\nu} - \frac{\omega^\mu \omega^\nu}{\omega^2}: \quad \omega_\mu N^{\mu\nu} = 0, \quad N^2 = N. \quad (5)$$

Let us construct Hamiltonian formulation of the model. We denote the conjugated momenta for ω^μ as π_μ , and the conjugated momenta for x^μ by p_μ . Applying the standard machinery [14, 15] we get that the system possesses

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two primary constraints

$$\omega\pi = 0, \quad \omega^2\pi^2 + c_1^2 = 0, \quad (6)$$

and the Hamiltonian

$$H = p\omega - c_0|\omega| + v_1(\omega^2\pi^2 + c_1^2) + v_2\omega\pi, \quad (7)$$

where v_i are Lagrangian multipliers associated with the primary constraints. Note, that Eq. (6) together with $\omega^2 < 0$ imply $\pi^2 > 0$.

Combining the constraints, we could replace $\omega^2\pi^2 = -c_1^2$ by the equivalent constraint $J^{\mu\nu}J_{\mu\nu} = -8c_1^2$, where $J^{\mu\nu} = 2\omega^{[\mu}\pi^{\nu]}$ is angular momentum of (ω, π) -space. As each motion take place with the same angular momentum, we expect that quantum theory describes an elementary system of fixed spin.

Computing derivatives of the primary constraints, and so on, we get the following sets of the first-class constraints

$$\omega\pi = 0, \quad \Rightarrow \quad p\omega - c_0|\omega| = 0, \quad (8)$$

the second-class ones

$$\omega^2\pi^2 + c_1^2 = 0, \quad \Rightarrow \quad p\pi = 0, \quad (9)$$

as well as the equation determining the Lagrangian multiplier, $v_1 = -|\omega|\frac{p^2+c_0^2}{2c_0c_1^2}$. The multiplier v_2 remains an arbitrary function, in accordance with reparametrization invariance of the action (1).

Then we can immediately compute the number of physical degrees of freedom in the spin sector parameterized by 8 phase-space variables ω^μ, π_ν subject to two second-class constraints written in Eq. (9), and two first-class constraints (8): $8 - (2 + 2 \times 2) = 2$, as it should be.

To take into account the second-class constraints (9), we pass from Poisson to Dirac bracket, given by the relations

$$\begin{aligned} \{\omega^\mu, \omega^\nu\}_D &= \frac{\omega^2}{\pi^2(p\omega)} p^{[\mu}\pi^{\nu]}, \\ \{\omega^\mu, \pi^\nu\}_D &= \eta^{\mu\nu} - \frac{p^\mu\omega^\nu}{p\omega}, \quad \{\pi^\mu, \pi^\nu\}_D = 0. \end{aligned} \quad (10)$$

So, we are ready to quantize our model.

III. QUANTIZATION

Now we can quantize the Dirac brackets (10) and impose the first-class constraints (8) as operator equations on quantum states, thus obtaining some equations for the wave function. Before doing this, we pass from the variables ω, π to some another set, and show that quantization of the latter admit a reasonable interpretation of the resulting quantum theory as a model of spin one-half particle.

Namely, we note that in the model there is the set of phase-space functions

$$\tilde{J}^{5\mu} \equiv 2c_1 K_\alpha^\mu \left(\frac{\omega^\alpha}{|\omega|} + \frac{p^\alpha}{c_0} \right), \quad (11)$$

$$\tilde{J}^{\mu\nu} \equiv 2K_\alpha^\mu \left(\omega^\alpha + \frac{|\omega|p^\alpha}{c_0} \right) \pi^\beta K_\beta^\nu - (\mu \leftrightarrow \nu), \quad (12)$$

where

$$K_\alpha^\mu = \delta_\alpha^\mu - A p^\mu p_\alpha, \quad A = \frac{\sqrt{p^2 + c_0^2} - |c_0|}{p^2 \sqrt{p^2 + c_0^2}}. \quad (13)$$

Their Dirac brackets generate, on the first-class constraint surface, the $so(3,2)$ -algebra

$$\begin{aligned} \{\tilde{J}^{5\mu}, \tilde{J}^{5\nu}\} &= -2\tilde{J}^{\mu\nu}, \\ \{\tilde{J}^{\mu\nu}, \tilde{J}^{5\alpha}\} &= 2\eta^{\mu\alpha} \tilde{J}^{5\nu} - 2\eta^{\nu\alpha} \tilde{J}^{5\mu}, \\ \{\tilde{J}^{\mu\nu}, \tilde{J}^{\alpha\beta}\} &= 2(\eta^{\mu\alpha} \tilde{J}^{\nu\beta} - \eta^{\mu\beta} \tilde{J}^{\nu\alpha} - \eta^{\nu\alpha} \tilde{J}^{\mu\beta} + \eta^{\nu\beta} \tilde{J}^{\mu\alpha}). \end{aligned} \quad (14)$$

To see that this is indeed $so(3,2)$, let us denote $\tilde{J}^{AB} = (\tilde{J}^{5\mu}, \tilde{J}^{\mu\nu})$ and introduce the five-dimensional metric $\eta^{AB} = (-, +, +, +, -)$. Then the algebra (14) acquires the form

$$\{\tilde{J}^{AB}, \tilde{J}^{CD}\} = 2(\eta^{AC} \tilde{J}^{BD} - \eta^{AD} \tilde{J}^{BC} - \eta^{BC} \tilde{J}^{AD} + \eta^{BD} \tilde{J}^{AC}), \quad (15)$$

which is just $so(3,2)$ -algebra.

The first-class constraints (8) can be presented through the new variables as follows

$$p_\mu \tilde{J}^{5\mu} + 2c_1 \sqrt{p^2 + c_0^2} = 0, \quad (16)$$

$$(\omega\pi)^2 = -c_1^2 + \frac{1}{p^2} \left(\frac{1}{4} S^\mu S_\mu - c_1^2 c_0^2 \right), \quad (17)$$

where $S^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} p_\nu \tilde{J}_{\alpha\beta}$ is Casimir operator of Poincare group (Pauli-Lubanski vector).

The initial coordinates x^μ have non-zero Dirac brackets with \tilde{J}^{AB} . To improve this, we introduce the effective coordinates $X^\mu = x^\mu + |w|\pi^\mu/c_0$ commuting with \tilde{J}^{AB} , and take operators associated with X^μ, p_μ in the standard form, $\hat{X}^\mu = X^\mu, \hat{p}_\mu = -i\hbar \frac{\partial}{\partial X^\mu}$.

To quantize the spin variables, we look for operators with commutators obeying the Dirac-brackets algebra on the constraint surface, the equation (14). That is we adopt the rule $[\ , \] = i\hbar \{ \ , \ }_{DB}|_{1CC}|_{q \rightarrow \hat{q}}$. This guarantees the correspondence principle: the operators (in the Heisenberg picture) and corresponding classical variables will obey the same equations of motion.

Choice of spin-sector operators is dictated by the constraint $\omega\pi = 0$ as follows. According to Eq. (10), this contains product of variables with non-vanishing Dirac brackets, so the corresponding quantum operator will contain product of non-commuting operators. Any

two (Lorentz-invariant) operators which we can associate with the constraint differ on a number. So, there is an ambiguity in the passage from classical to quantum theory

$$\omega\pi = 0 \rightarrow \hat{\omega}\hat{\pi} = c_2. \quad (18)$$

We propose to fix the ordering constant to be $c_2^2 = -c_1^2$. Then quantum counterpart of equation (17) states that Casimir operator of the Poincaré group has fixed value

$$S^2 = 4c_1^2 c_0^2 = 3\hbar^2 m^2 c^2, \quad (19)$$

corresponding to spin one-half irreducible representation (Note that our state-space is picked out by unique equation (16). The only irreducible representation of Poincaré group of such a kind is the spin one-half representation. So any choice of c_1 different from $c_1 = \frac{\sqrt{3}\hbar}{2}$ would lead to inconsistent picture).

Hence we are forced to quantize the variables $\tilde{J}^{5\mu}$ and $\tilde{J}^{\mu\nu}$ by γ -matrices, $\tilde{J}^{5\mu} \rightarrow \hbar\gamma^\mu$, $\tilde{J}^{\mu\nu} \rightarrow \hbar\gamma^{\mu\nu}$, where

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad (20)$$

and

$$\gamma^{\mu\nu} \equiv \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu). \quad (21)$$

They form a representation of $so(3,2)$

$$\begin{aligned} [\gamma^\mu, \gamma^\nu] &= -2i\gamma^{\mu\nu}, & [\gamma^{\mu\nu}, \gamma^\alpha] &= 2i(\eta^{\mu\alpha}\gamma^\nu - \eta^{\nu\alpha}\gamma^\mu), \\ [\gamma^{\mu\nu}, \gamma^{\alpha\beta}] &= 2i(\eta^{\mu\alpha}\gamma^{\nu\beta} - \eta^{\mu\beta}\gamma^{\nu\alpha} - \eta^{\nu\alpha}\gamma^{\mu\beta} + \eta^{\nu\beta}\gamma^{\mu\alpha}) \end{aligned} \quad (22)$$

The operators act on space of Dirac spinors Ψ_a , $a = 1, 2, 3, 4$.

Let us see the meaning of the first-class constraint (16). Its quantum counterpart reads

$$\left(\gamma^\mu \hat{p}_\mu + \frac{2c_1}{\hbar} \sqrt{\hat{p}^2 + c_0^2} \right) \Psi = 0. \quad (23)$$

Then we obtain, as a consequence, the following Klein-Gordon equation

$$\left(\hat{p}^2 + \frac{4c_0^2 c_1^2}{\hbar^2 + 4c_1^2} \right) \Psi = 0. \quad (24)$$

Substitute (24) into (23), this gives the Dirac equation $(\gamma^\mu \hat{p}_\mu - 2c_0 c_1 (\hbar^2 + 4c_1^2)^{-\frac{1}{2}}) \Psi = 0$. Taking $c_0 = -mc$ and $c_1 = \frac{\sqrt{3}\hbar}{2}$, we arrive at the final result

$$\left(\gamma^\mu \hat{p}_\mu + \frac{\sqrt{3}}{2} mc \right) \Psi = 0. \quad (25)$$

In resume, canonical quantization of the action (1) in properly chosen variables (11), (12) leads to the theory of spin one-half particle. This observation contradicts with common opinion about role of higher-derivative models in the description of massive spinning particles. Many questions arise in this respect. Could other models with Lagrangians depending on extrinsic curvatures (say, on torsion) describe massive and massless spinning particles on the non-constant curvature space-time? If so, of which kind, and of which spin? Could this revision changes our opinion on the role of such systems in quantum optics and polymer physics? Clarification of these questions should be subject of separate study.

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